TECHNISCHE UNIVERSITÄT DRESDEN

Institute for Static und Dynamics of Structures

Mathematical Basics of Fuzziness

Bernd Möller

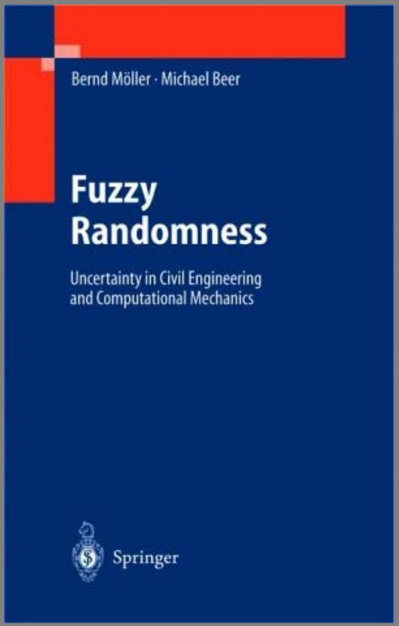
Recent research results about
non-classical methods in uncertainty modeling

http://www.uncertainty-in-engineering.net

and

Fuzzy Randomness

B. Möller, M. Beer,
Springer 2004



1 Introduction and motivation

2 Fuzzy variables

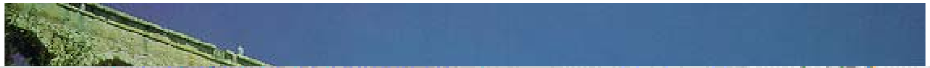
3 Fuzzy functions

4 Fuzzy analysis

5 Quantification of fuzziness

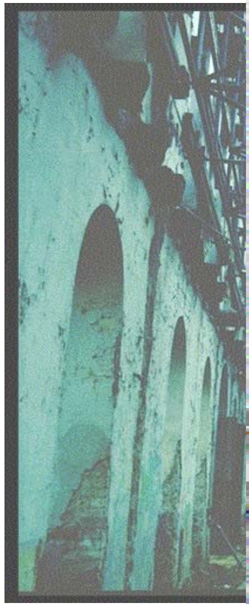
6 Assessment of fuzzy results

bridge engineering



underground engineering

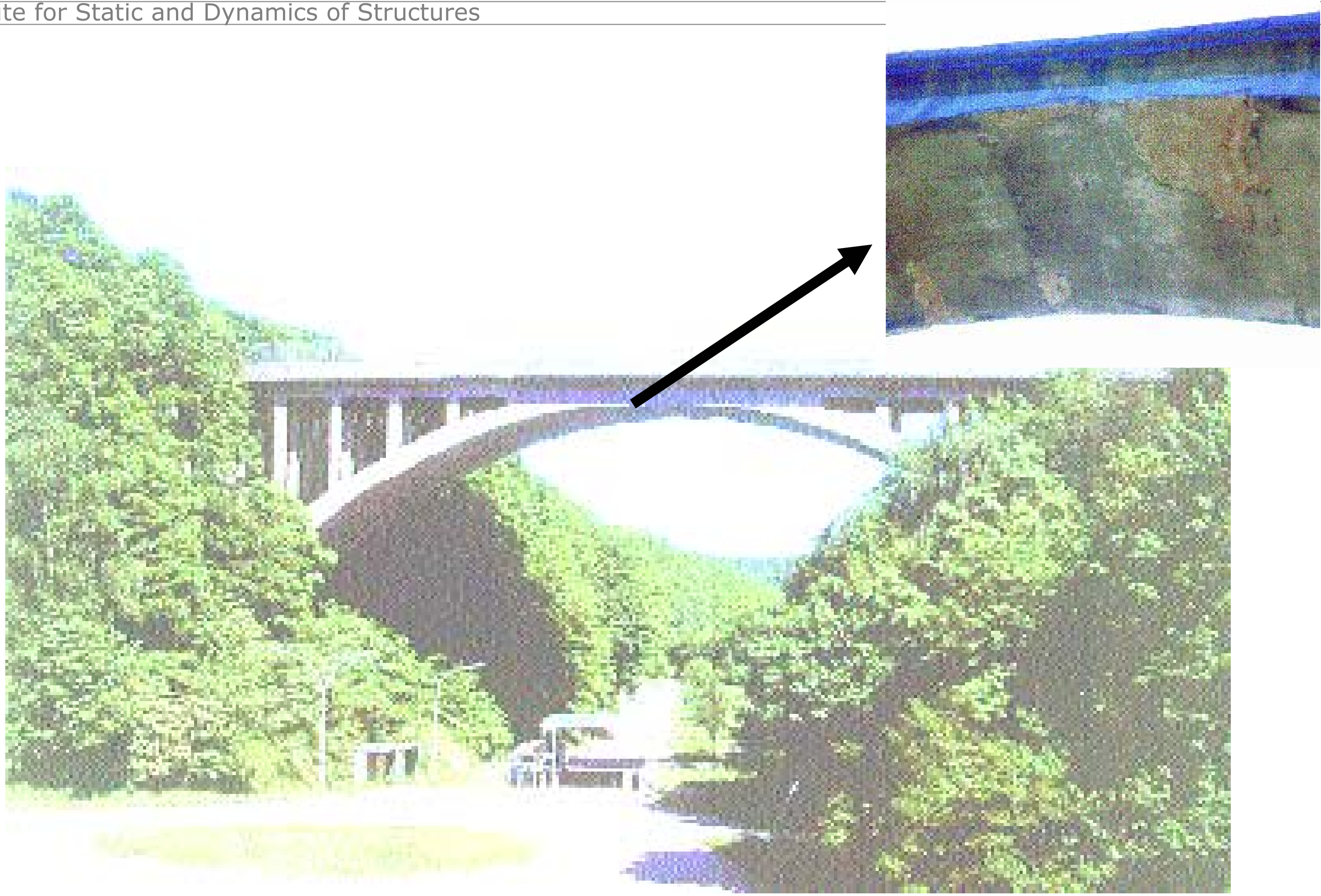
building construction



Prestressed Reinforced Concrete Trusses (prefabricated segments)

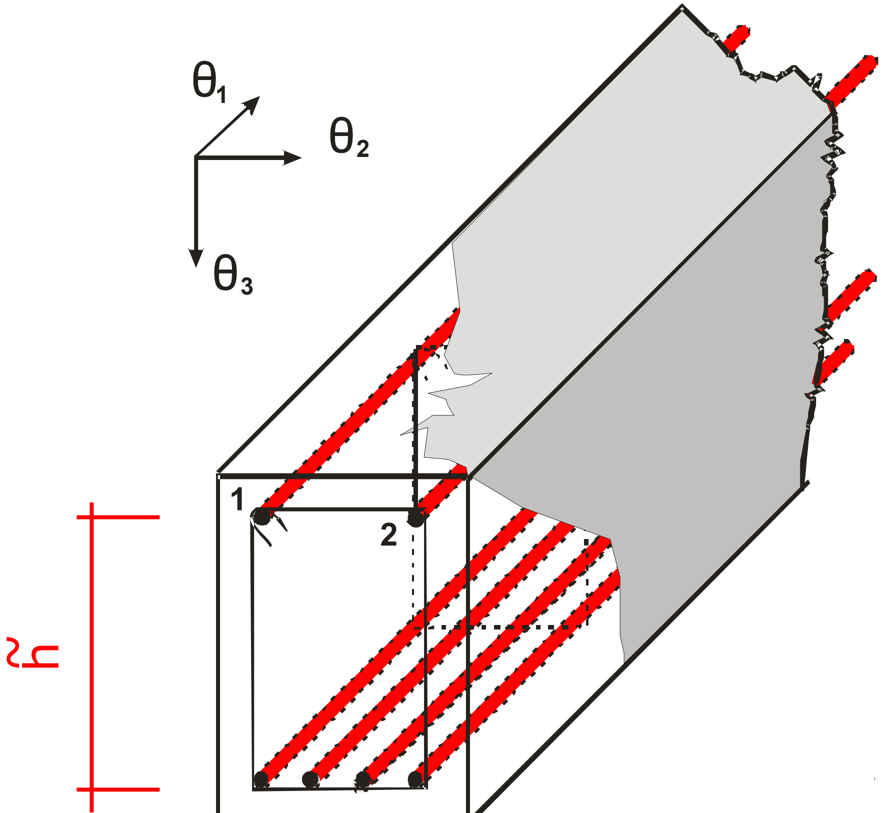
damage of concrete

Mathematical Basics - Fuzziness



deterioration of an arch brigde (Pirmasens, Germany)

uncertain geometrical parameters



uncertain position of reinforcement

Which structural parameters might be uncertain ?

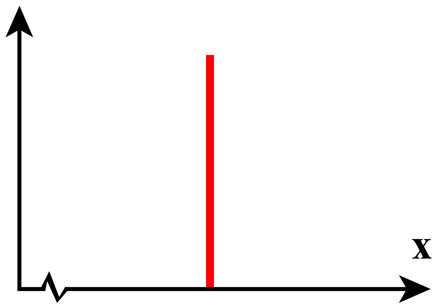
geometry ?

material ?

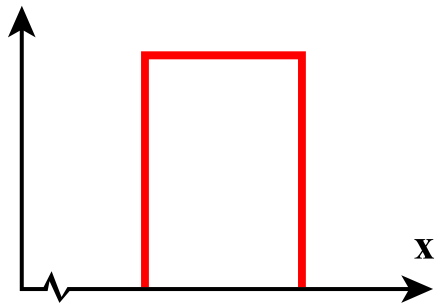
loading ?

foundation ?

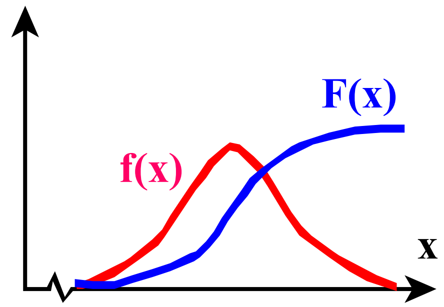
- **deterministic variable**



- **interval variable**



- **random variable**

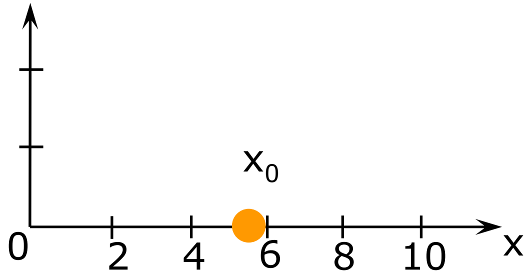


additional uncertainty models

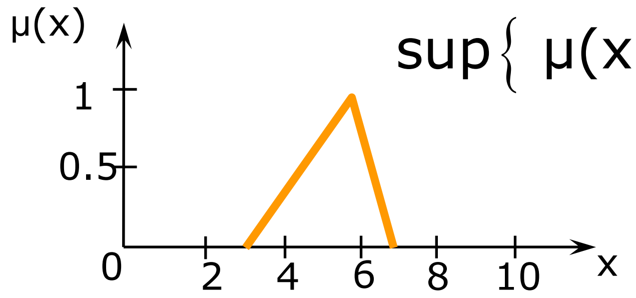
- **fuzzy variable**
- **fuzzy random variable**

Mathematical Basics - Fuzziness

deterministic variable

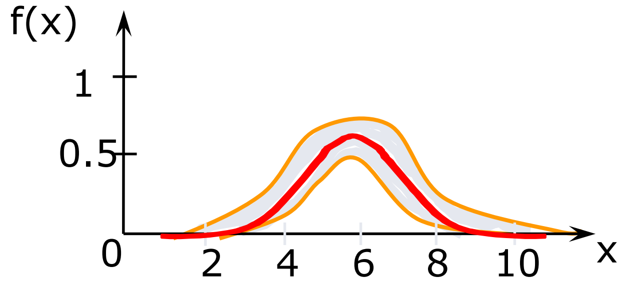


fuzzy variable

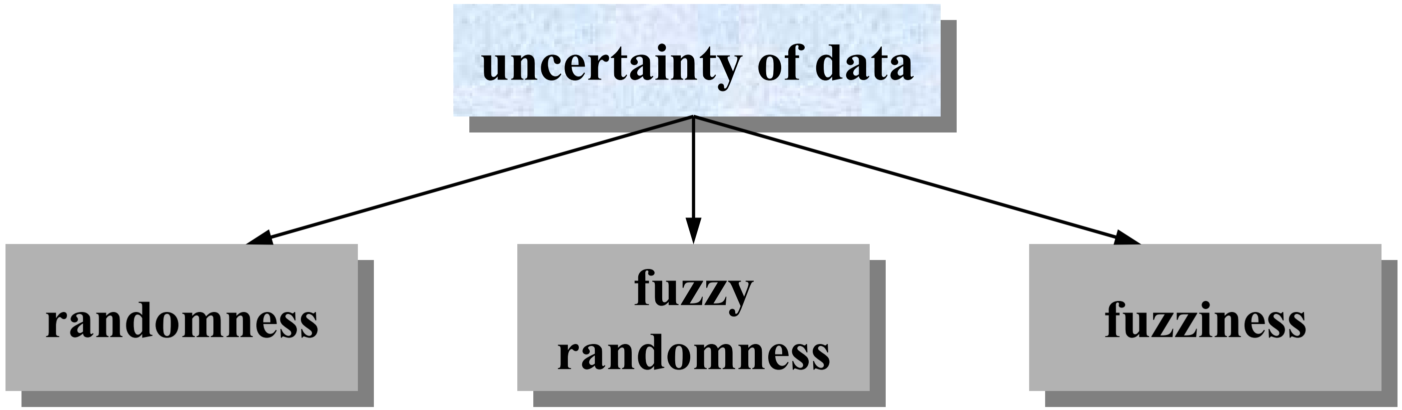


$$\sup\{\ \mu(x)\ |\ x \in R\} = 1$$

fuzzy random variable



Mathematical Basics - Fuzziness



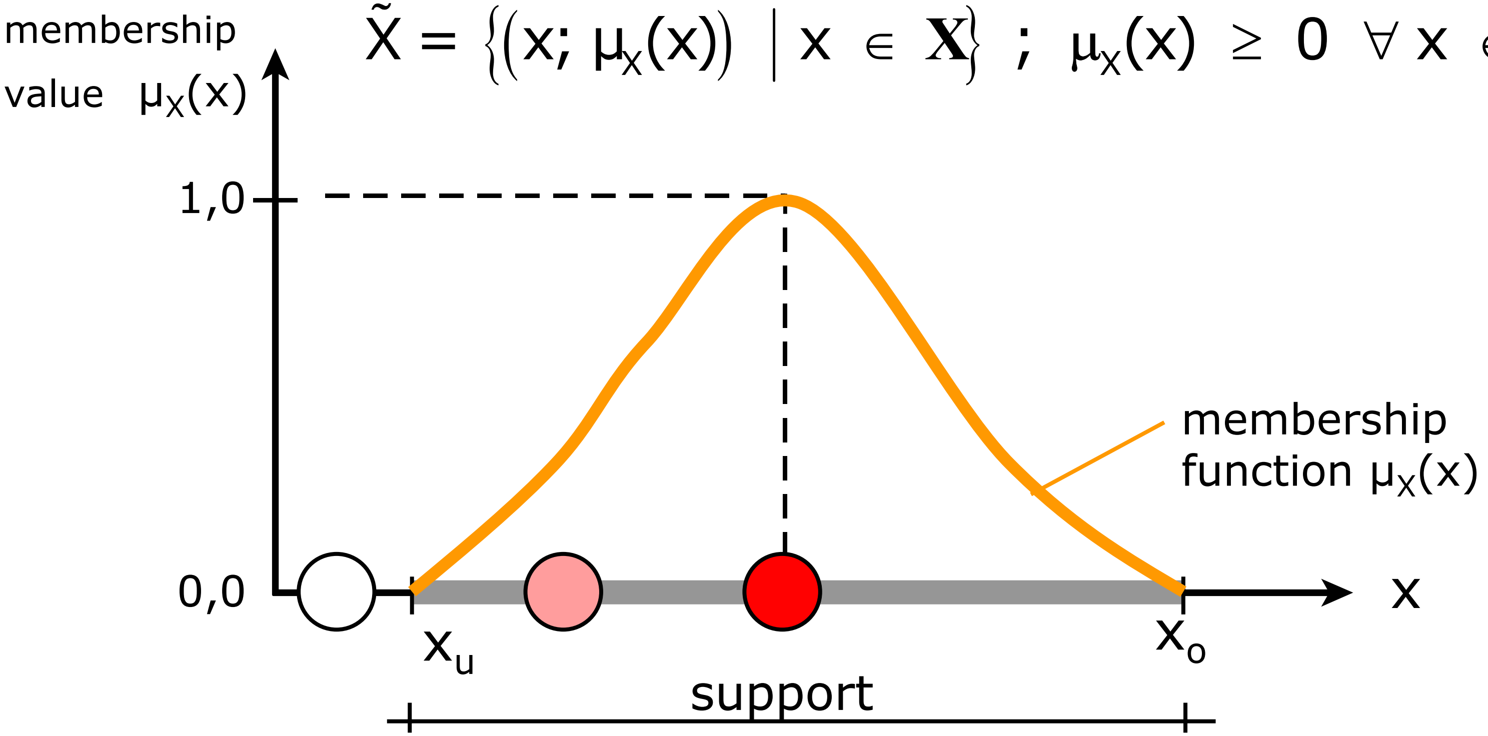
examples for uncertainties

- earthquake loading
- storm loading
- impact loads
- aging processes
- damage
- material parameters

Fuzzy variables

fuzzy variable $\tilde{X}$

$$\tilde{X} = \left\{ \left(x;\, \mu_X(x) \right) \,\middle|\, x \in \mathbf{X} \right\} \;;\; \mu_X(x) \geq 0 \;\; \forall\, x \in \mathbf{X}$$



fuzzy number: at exactly one x the membership $\mu_X(x) = 1$

Fuzzy variables

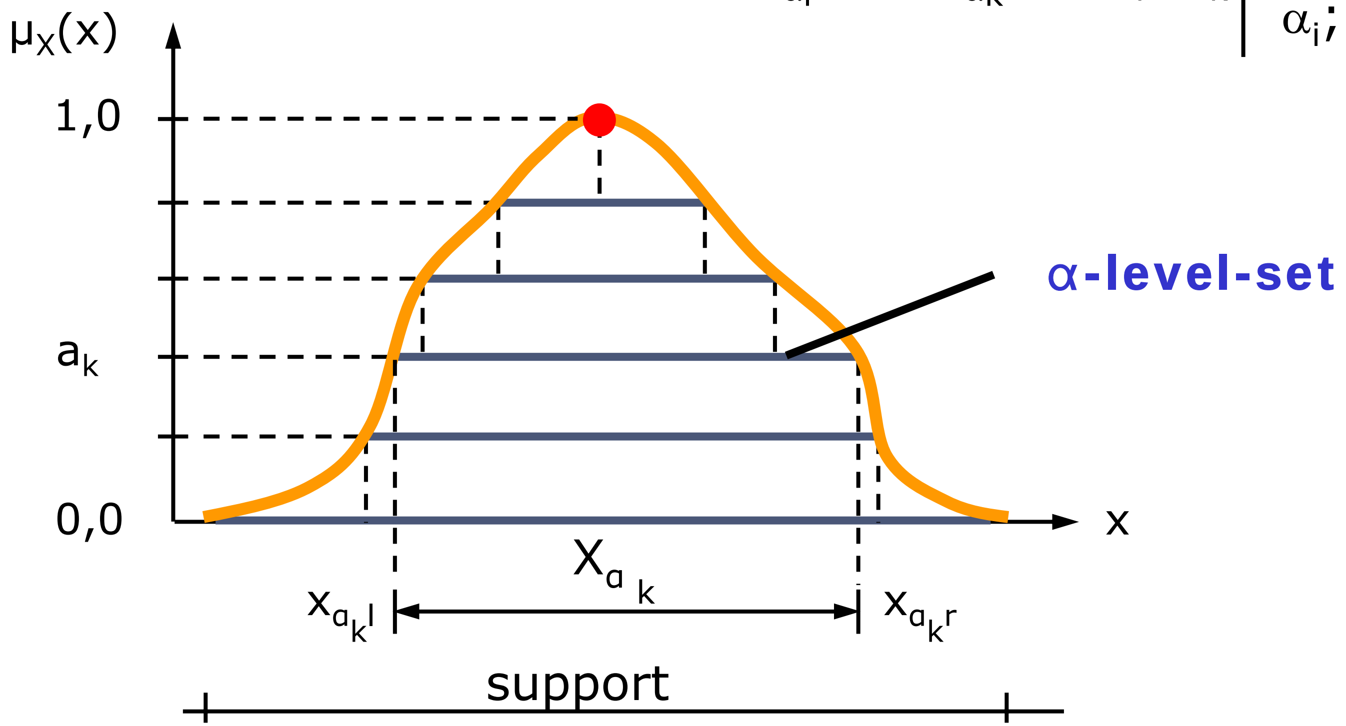
α-level-set

$$X_{\alpha_k} = \{ x \in X \mid \mu_x(x) \geq \alpha_k \}$$

α-discretization

$$\tilde{X} = \{ (X_\alpha ; \mu(X_\alpha)) \}$$

$$X_{\alpha i} \subseteq X_{\alpha k} \quad \forall \, \alpha_i, \alpha_k \,\Big|\, \begin{array}{l} \alpha_i \geq \alpha_k \\ \alpha_i ; \alpha_k \in (0,1] \end{array}$$



1 Introduction and motivation

2 Fuzzy variables

3 Fuzzy functions

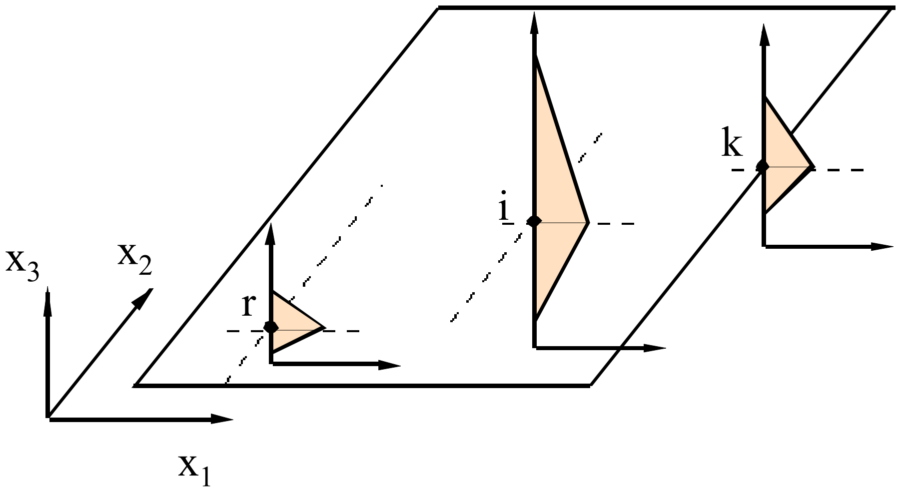
4 Fuzzy analysis

5 Quantification of fuzziness

6 Assessment of fuzzy results

Fuzzy function of the form : $\tilde{y} = \tilde{f}(\underline{x})$:

to every crisp point
an uncertain physical parameter may be assigned

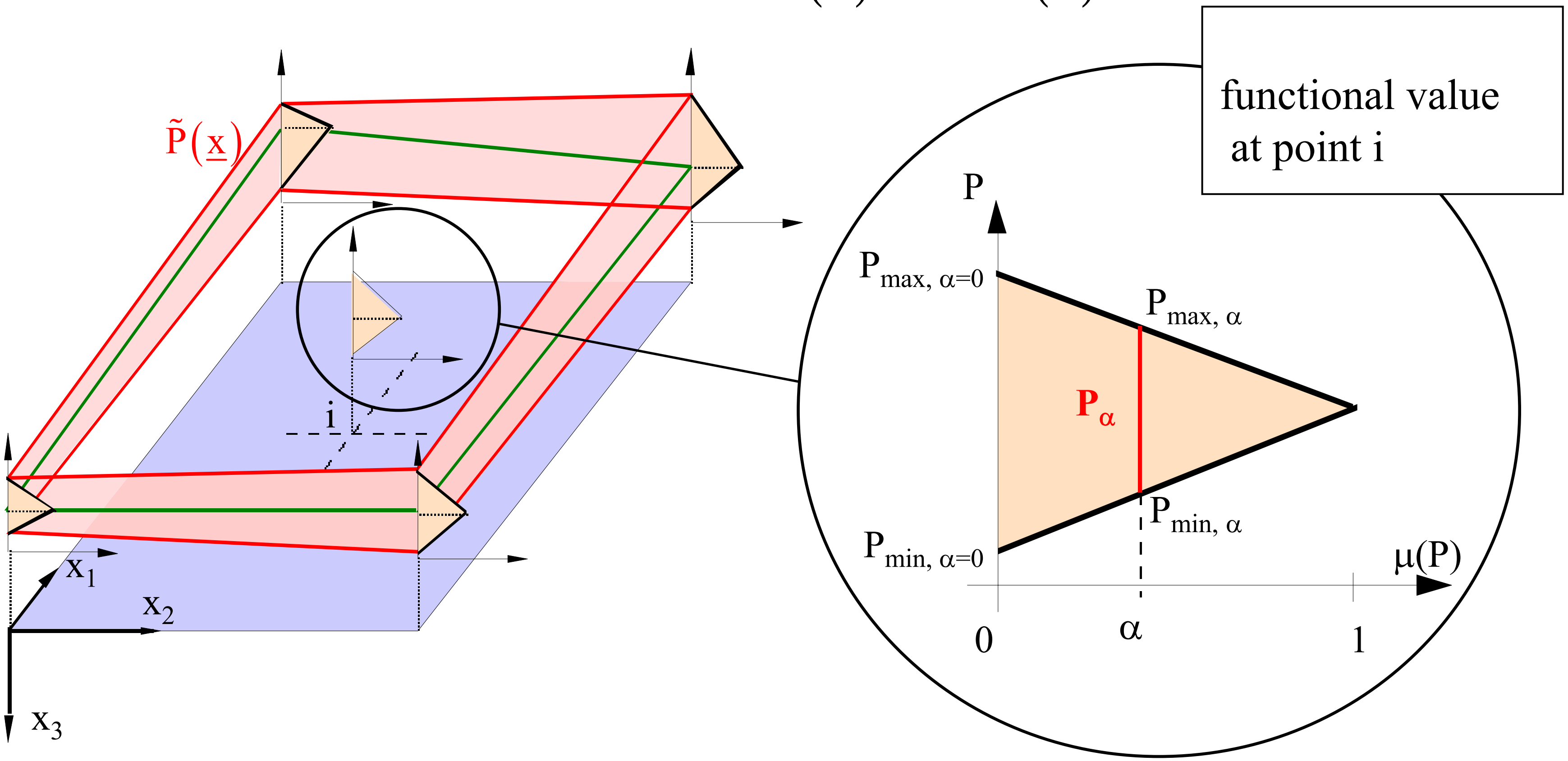


$\underline{\mathbf{x}} = \{\mathbf{x}_1, \mathbf{x}_2\}$ **in** $\mathbb{R}^1$ **and** $\mathbb{R}^2$

uncertain physical parameters at crisp points i, k, r in $\mathbb{R}^2$:

- damping coefficient
- boundary conditions
- loading
- material parameters
- position of reinforcement

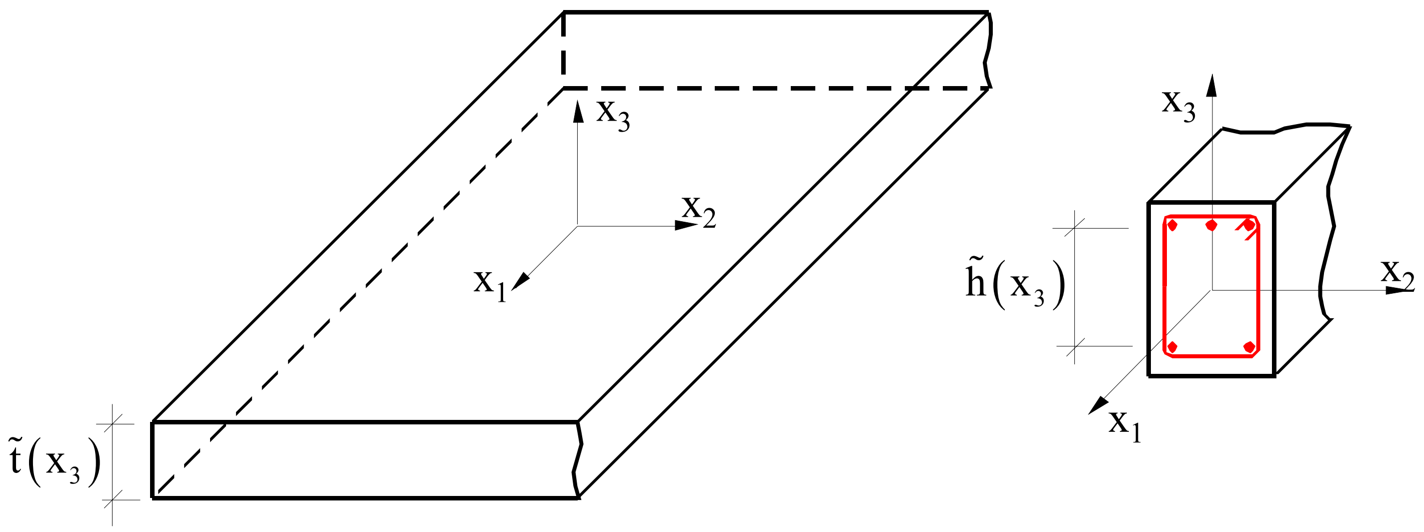
fuzzy function of physical parameter $\tilde{P}(\underline{x}) = \tilde{y} = \tilde{f}(\underline{x})$:



$$\tilde{P}(\underline{x}) = \left\{ P_\alpha(\underline{x});\ \mu\left(P_\alpha(\underline{x})\right) \middle| P_\alpha(\underline{x}) = \left[P_{min,\alpha}(\underline{x});\ P_{max,\alpha}(\underline{x})\right];\ \mu\left(P_\alpha(\underline{x})\right) = \alpha\ \forall\, \alpha = (0;1] \right\}$$

sometimes also geometrical parameters in x_3-direction are uncertain

example of structures in $\mathbb{R}^1$ and $\mathbb{R}^2$



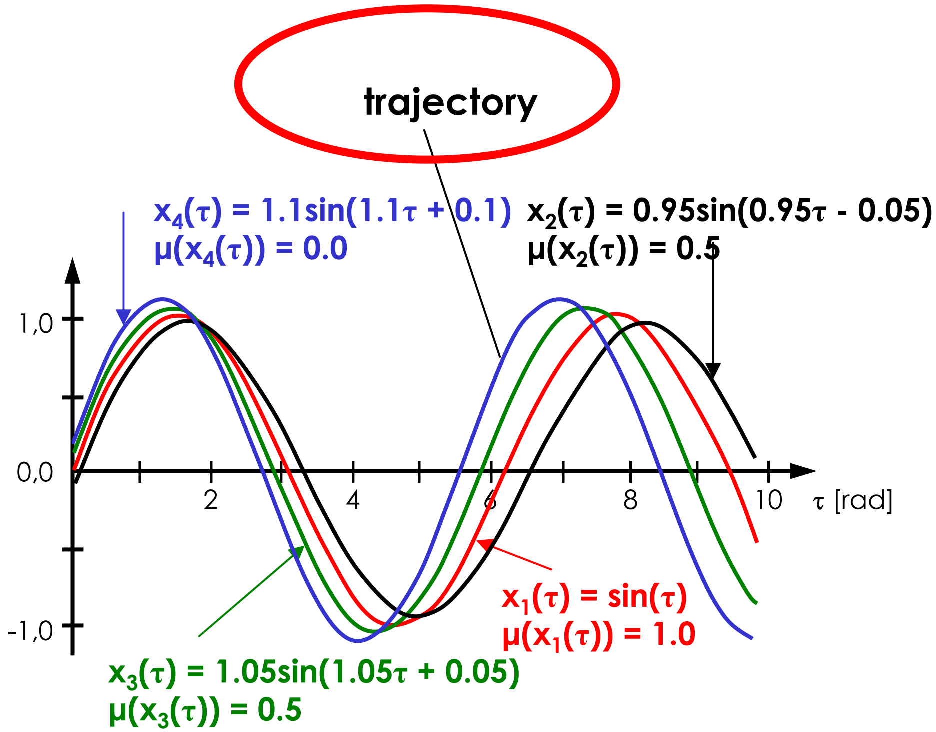
geometrical parameters dependent on x_3 may be expressed as fuzzy parameters

Bunch Parameter Representation of Fuzzy Functions

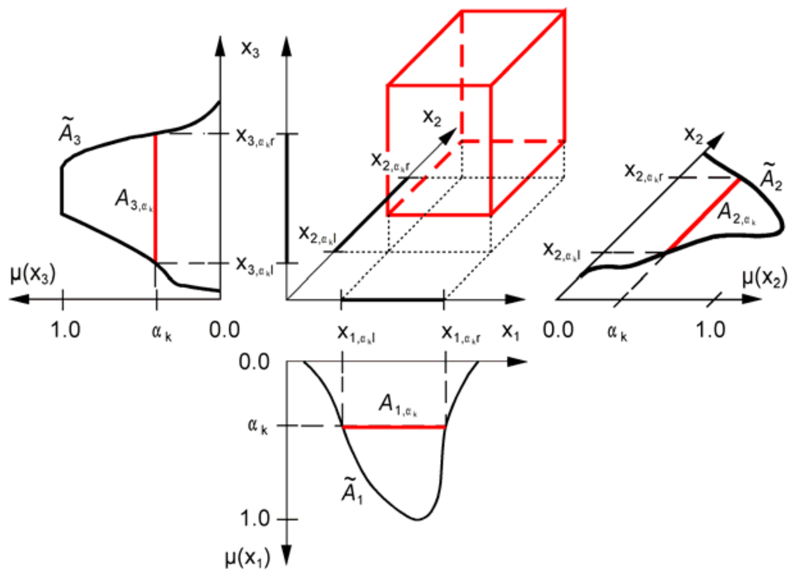
$\underline{\tilde{x}}(\underline{t}) = x(\underline{\tilde{s}}, \underline{t})$

$\underline{\tilde{x}}(\tau) = x(\underline{\tilde{s}}, \tau) = \tilde{s}_1 \cdot \sin(\tilde{s}_2 \cdot \tau + \tilde{s}_3)$ with $\underline{\tilde{s}} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$

$\tilde{s}_1 = < 0.9, 1.0, 1.1 >$
$\tilde{s}_2 = < 0.9, 1.0, 1.1 >$
$\tilde{s}_3 = < -0.1, 0.0, 0.1 >$

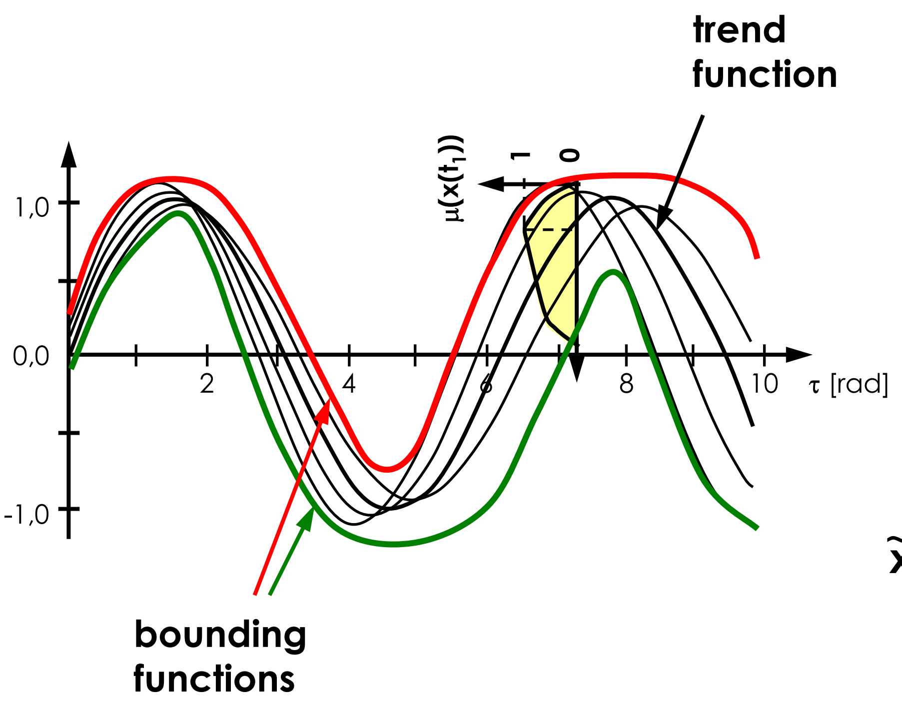


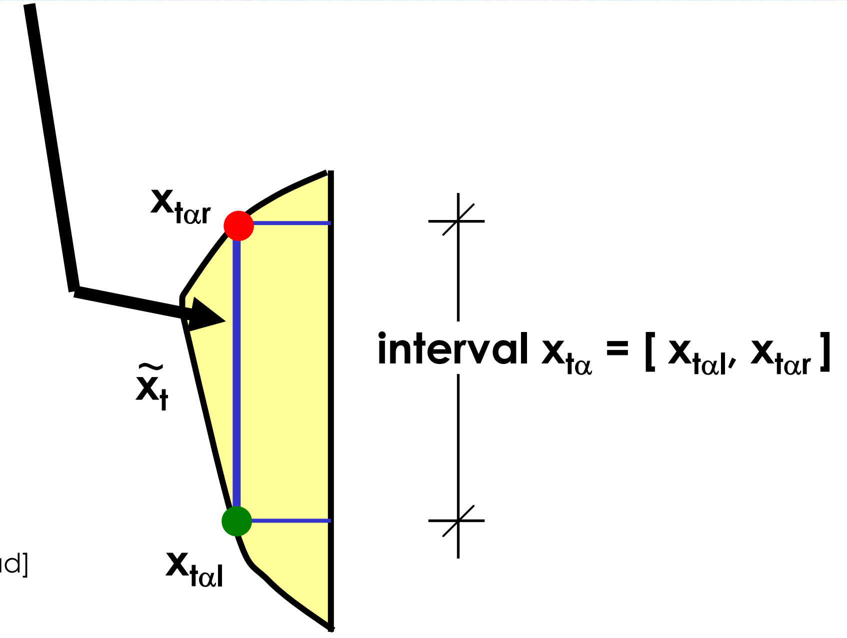
crisp subspace $\underline{S}_\alpha$ with $\underline{s} \in \underline{S}_\alpha$



Bunch Parameter Representation of Fuzzy Functions

For $\underline{s} \in \underline{S}_\alpha$ crisp set of functions : $X_\alpha(\underline{t}) = \{x(\underline{s},\underline{t})\ \ \forall \underline{s} \mid \underline{s} \in \underline{S}_\alpha;\ \alpha \in (0,1]\}$



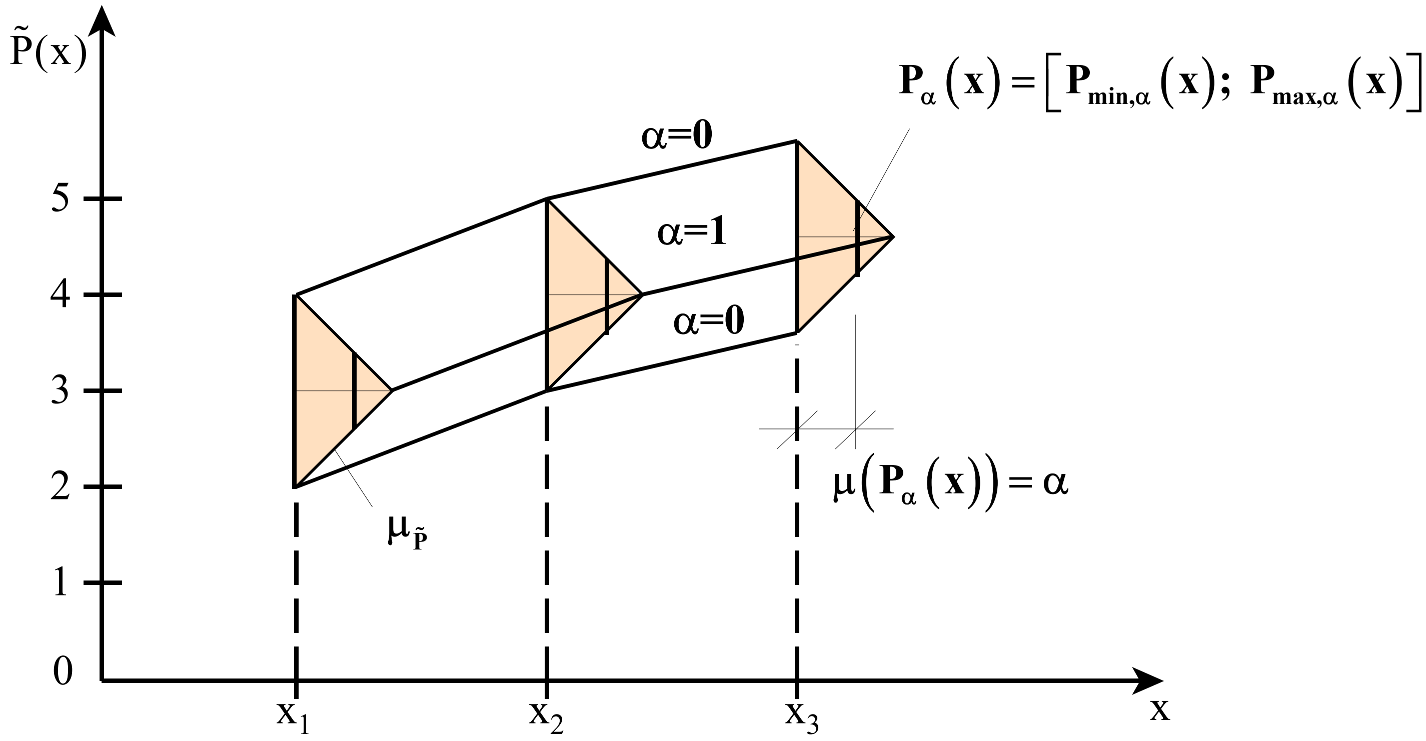


$$\tilde{x}_t = x_t(\tilde{\underline{s}}) = \Big\{ \big(x_{t\alpha},\ \mu(x_{t\alpha}) \big) \ \Big|\ x_{t\alpha r} = [\,x_{t\alpha l},\ x_{t\alpha r}\,];$$

$$\mu(x_{t\alpha}) = \alpha \quad \forall \alpha \in (0,1] \Big\}$$

approximation of fuzzy function $\tilde{P}(\underline{x})$ by means of fuzzy values at suitably distributed interpolation nodes in $\mathbb{R}^n$

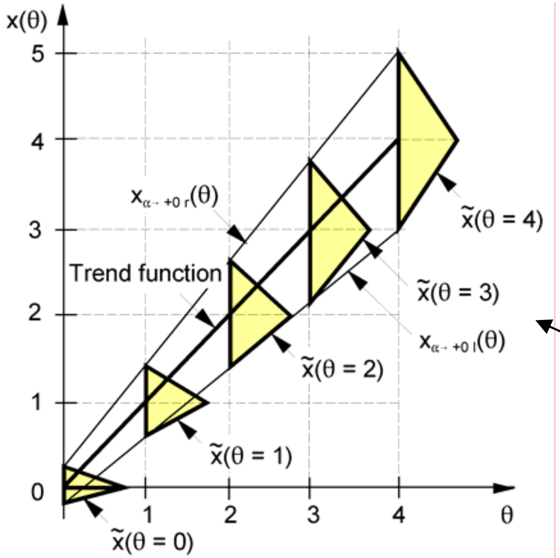
example:



approximation of the fuzzy function $\tilde{P}(\underline{x})$ in $\mathbb{R}^1$ using fuzzy numbers at x_1, x_2, x_3

Point and Time Discretization of Fuzzy Functions

The arguments $\underline{t}$ are discretized by k_0 interpolation nodes



for $\underline{t} = \theta$

$$\tilde{x}(\theta) = \tilde{s}_1 \cdot \theta + \tilde{s}_2$$

$$\tilde{s}_1 = < 0.8,\ 1.0,\ 1.2 >$$

$$\tilde{s}_2 = < -0.2,\ 0.0,\ 0.2 >$$

Interaction ?

Full interaction $\tilde{x}(\underline{t}) = \left\{ x(\underline{\tilde{s}},\underline{t}_1),\ x(\underline{\tilde{s}},\underline{t}_2),\ \ldots,\ x(\underline{\tilde{s}},\underline{t}_{k0}) \mid \underline{t}_1, \underline{t}_2, \ldots, \underline{t}_{k0} \in \underline{T} \right\}$

No interaction $\tilde{x}(\underline{t}) = \left\{ x(\underline{\tilde{s}}_1,\underline{t}_1),\ x(\underline{\tilde{s}}_2,\underline{t}_2),\ \ldots,\ x(\underline{\tilde{s}}_{k0},\underline{t}_{k0}) \mid \underline{t}_1, \underline{t}_2, \ldots, \underline{t}_{k0} \in \underline{T} \right\}$

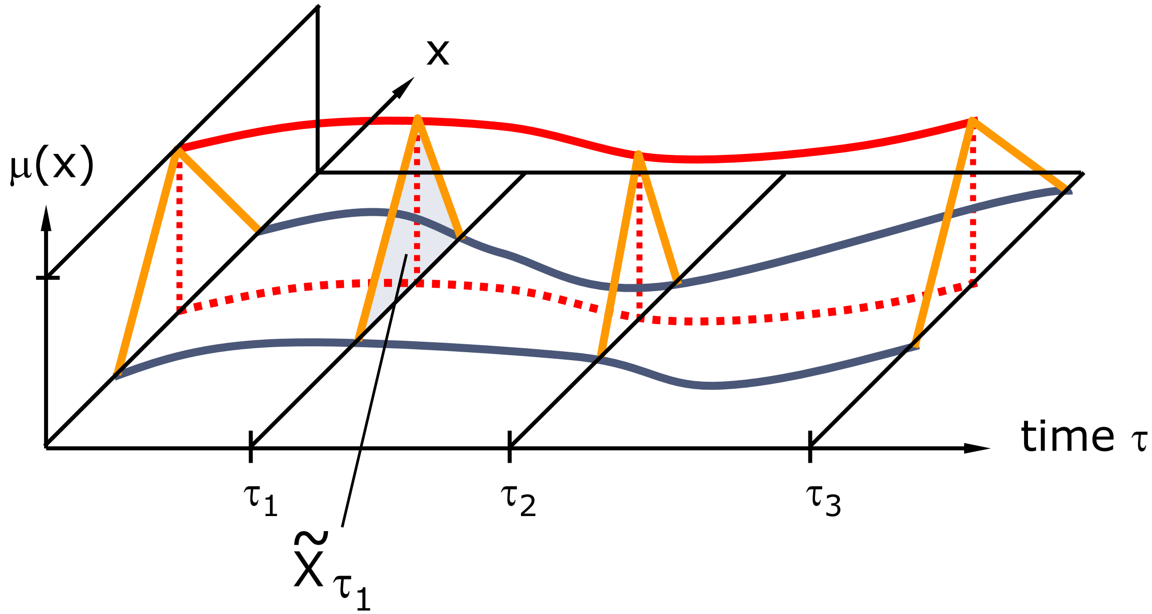
fuzzy function: $\tilde{\underline{x}}(\underline{t})$: T $\xrightarrow{\sim}$ F(X) $\underline{t} = (\underline{\theta}, \tau, \underline{\varphi})$

fundamental set (T)

set of fuzzy variables (F(X))

$$\tilde{\underline{x}}(\underline{t}) = \{\tilde{x}_t = \tilde{x}(\underline{t}) \ \forall \ t \mid t \in T\}$$

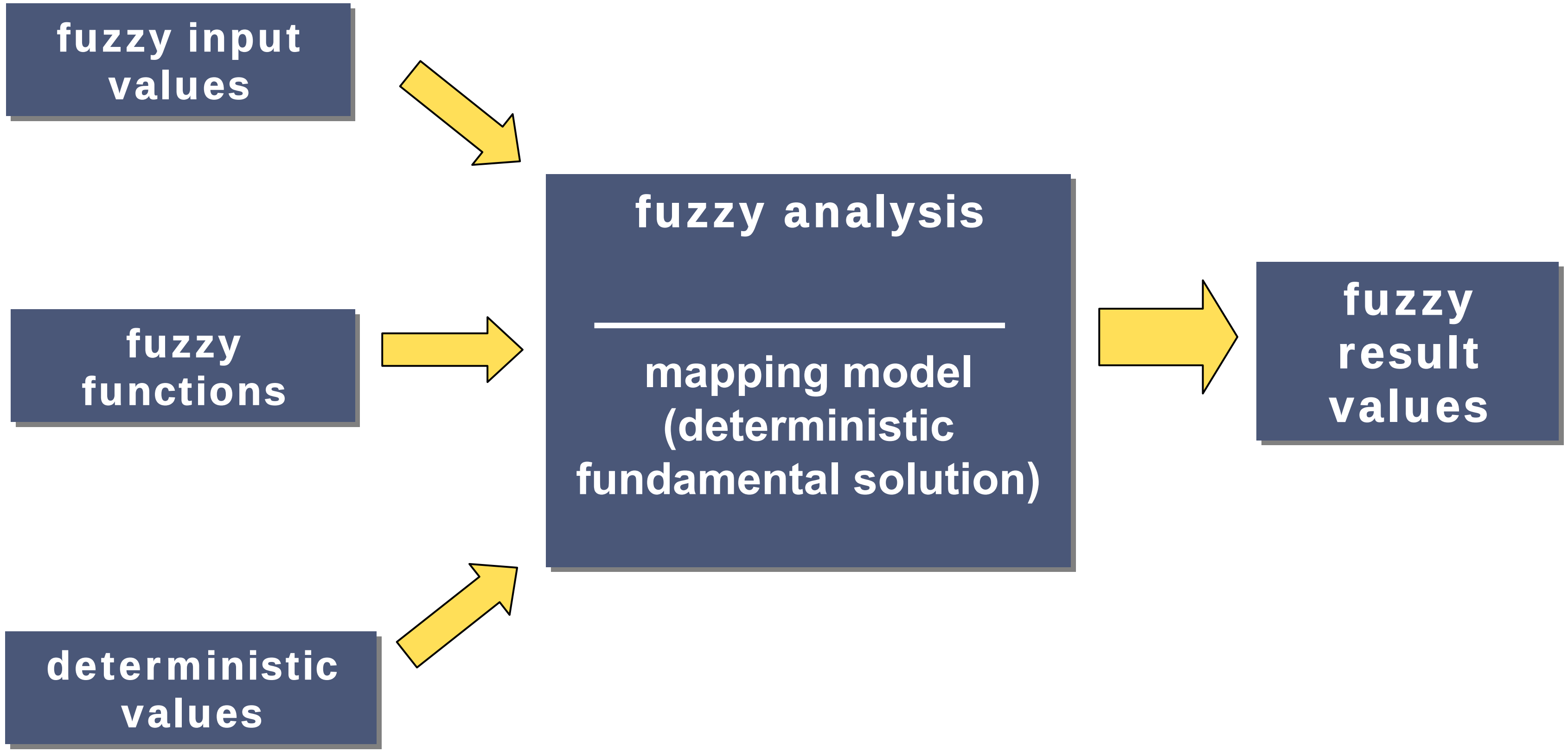
fuzzy process: $\tilde{\underline{x}}(\tau) = \{\tilde{x}_\tau = \tilde{x}(\tau) \ \forall \ \tau \mid \tau \in T\}$



see file 1.1 for
Cartesian product and extension principle

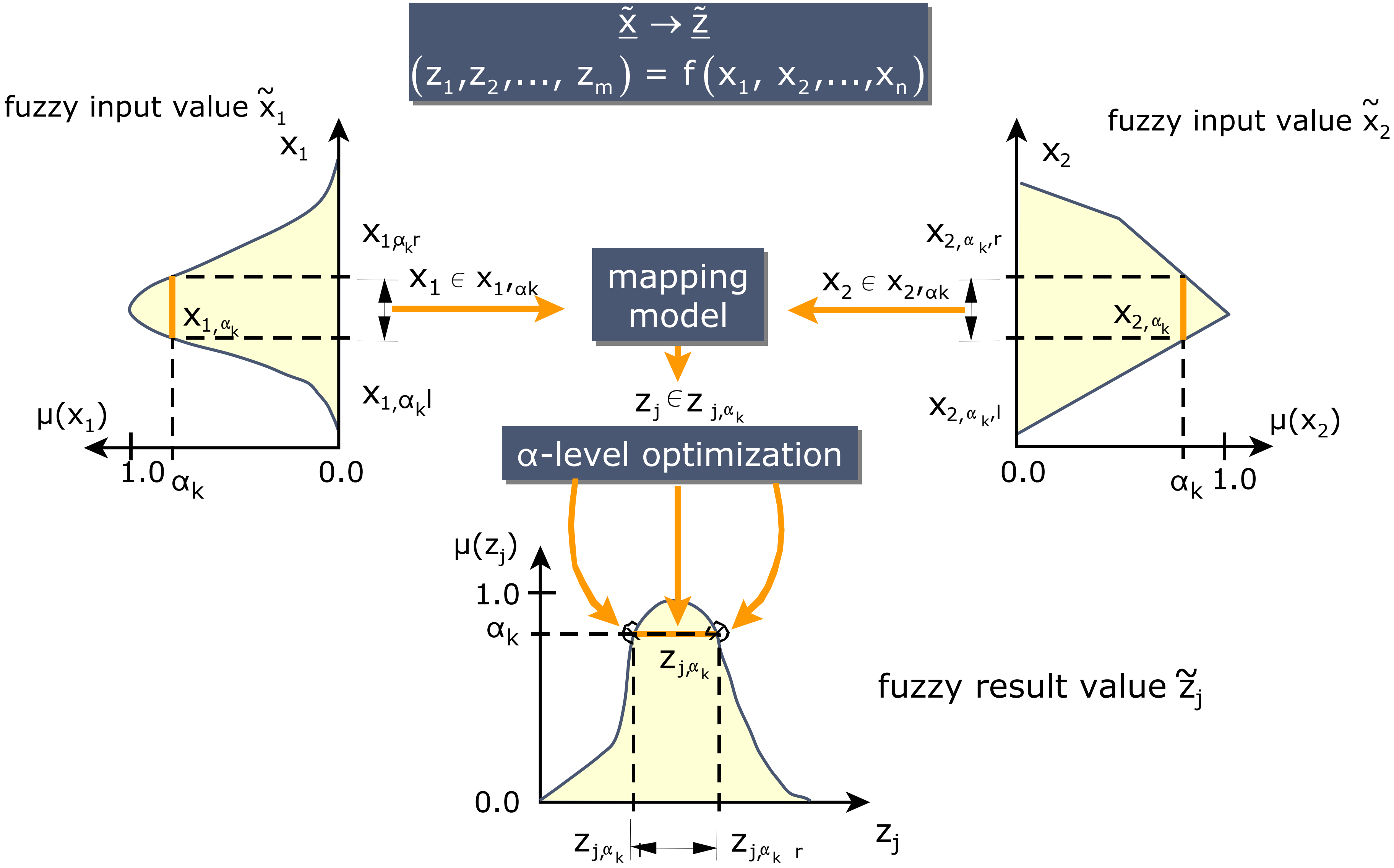
The Cartesian product comprises all combinations of the elements $x_1, \ldots\ldots, x_n$ **n** fuzzy sets.

Fuzzy analysis (1)

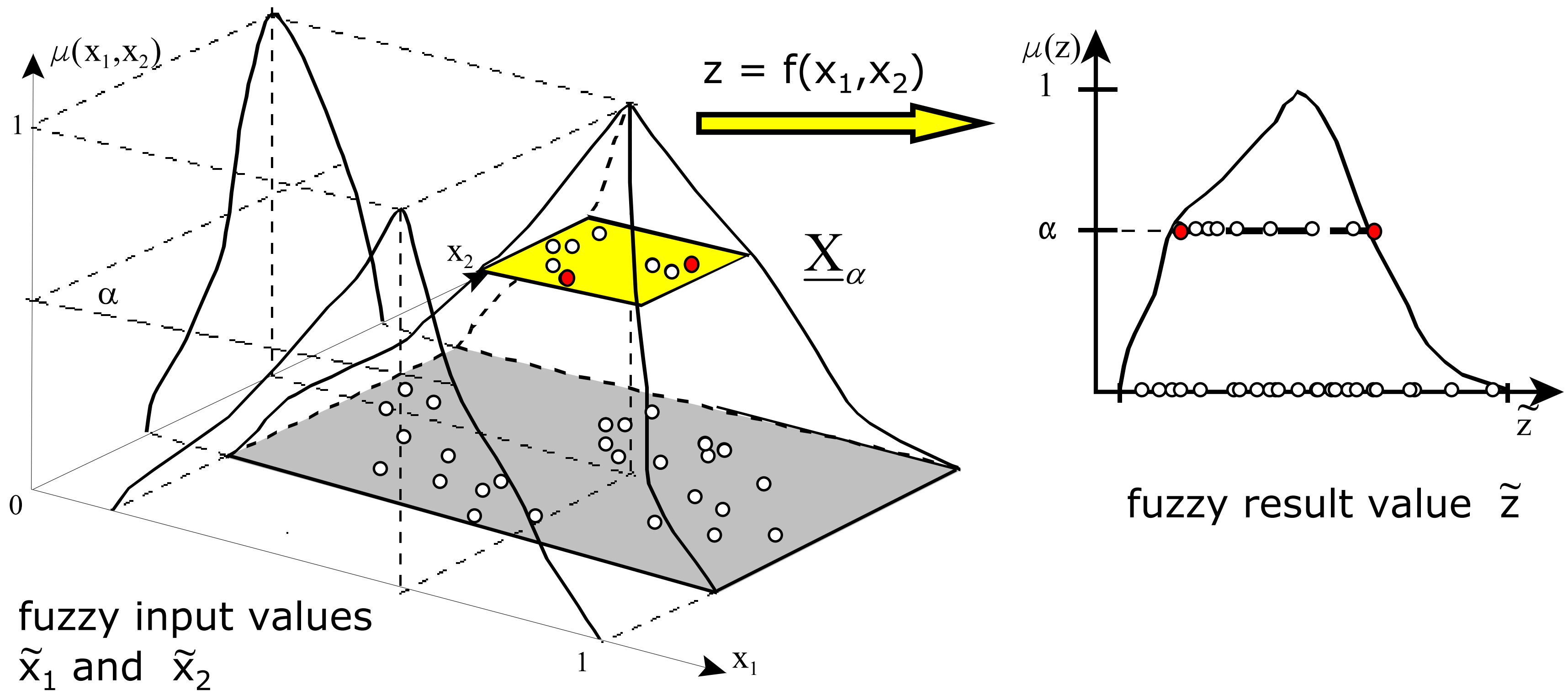


$$\underline{\tilde{X}}(\underline{\tilde{t}}) \xrightarrow{\sim} \underline{\tilde{Z}}(\underline{\tilde{t}})$$

Fuzzy analysis (2)



Fuzzy analysis (3)



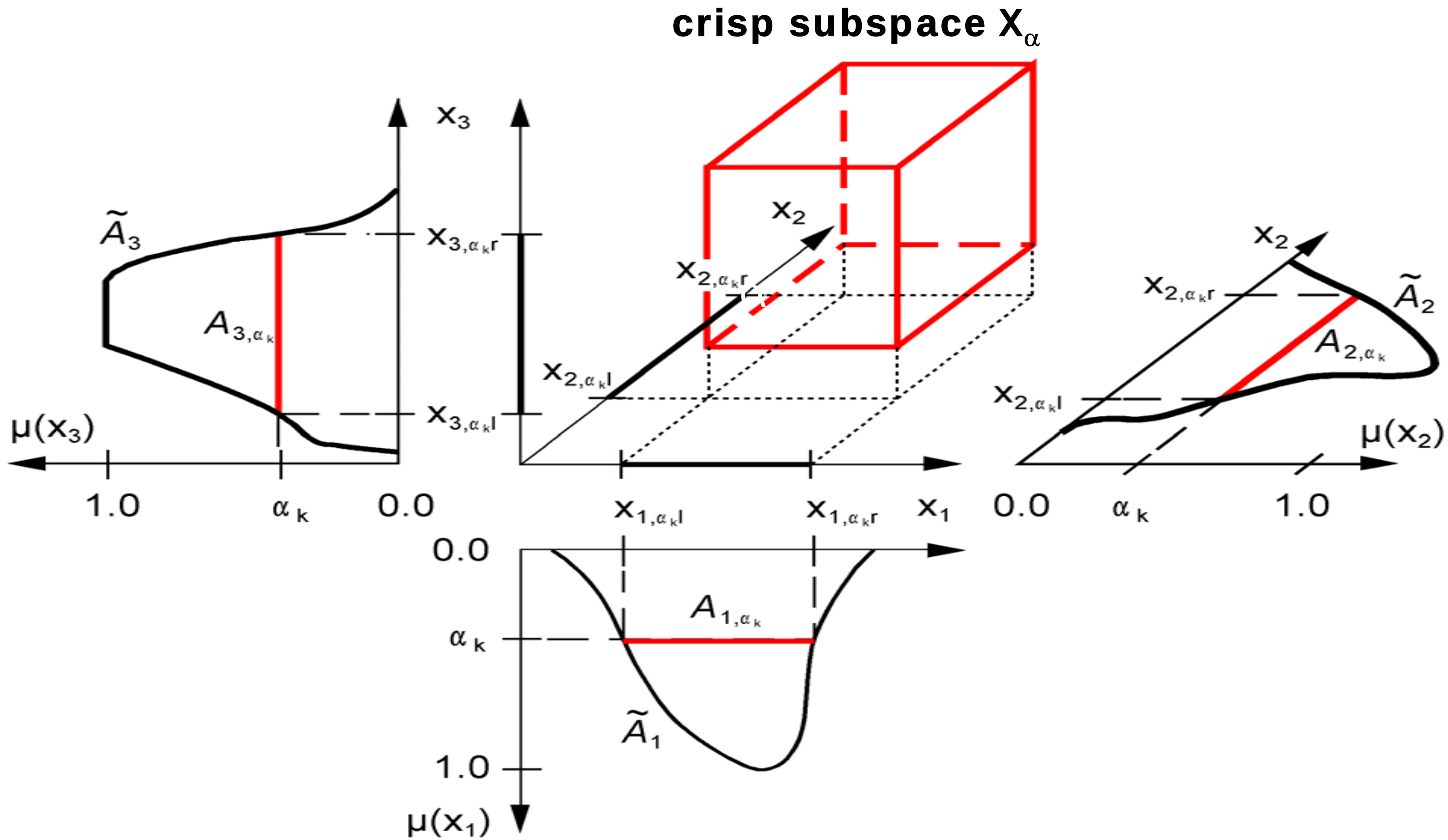
- objectiv functions: $z_j = f_j\left(x_1; \ldots; x_n\right) \;\Rightarrow\; \text{Max} \mid \left(x_1; \ldots; x_n\right) \in \underline{X}_\alpha$

  $z_j = f_j\left(x_1; \ldots; x_n\right) \;\Rightarrow\; \text{Min} \mid \left(x_1; \ldots; x_n\right) \in \underline{X}_\alpha$

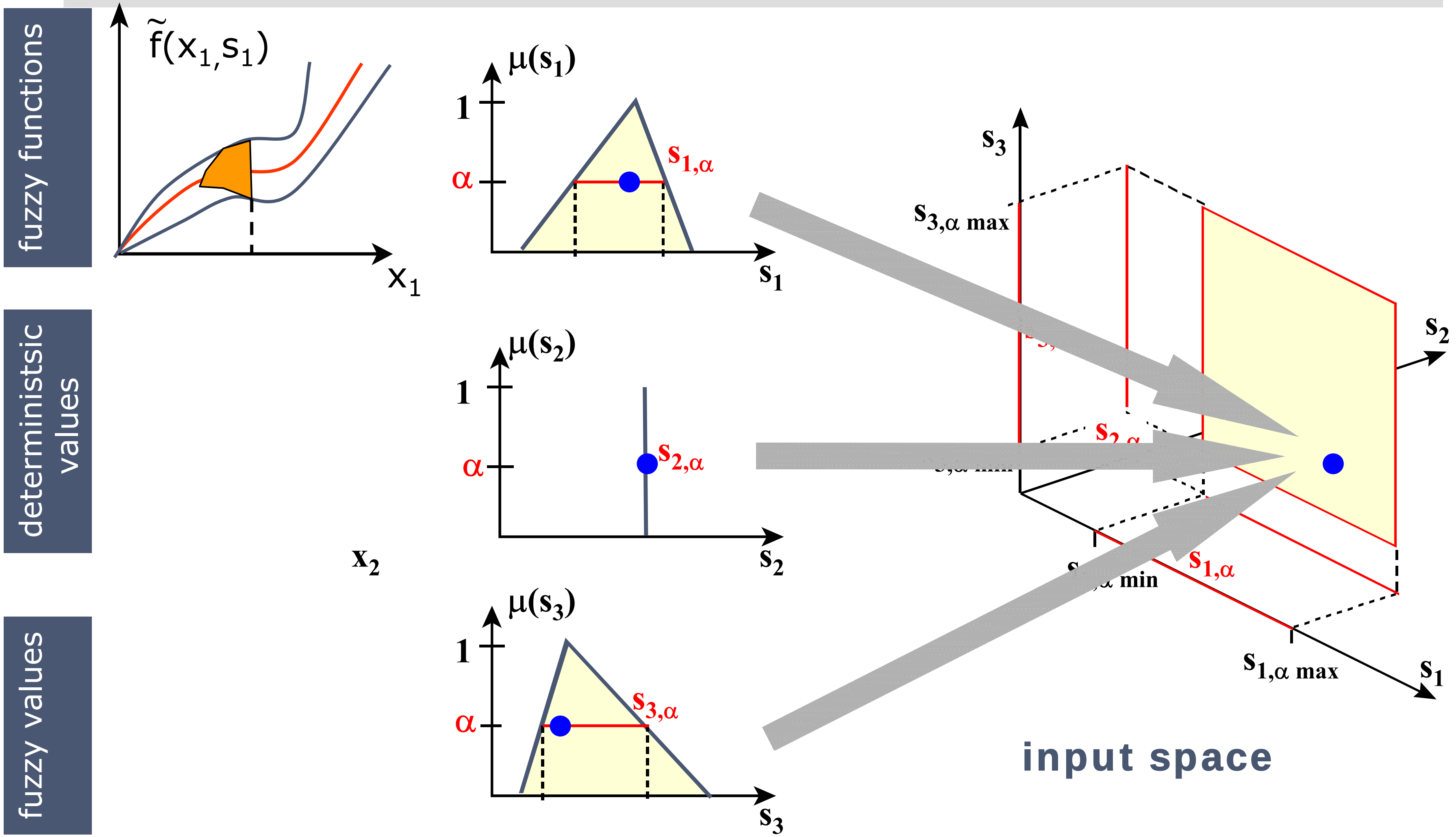
Fuzzy analysis (4)

constitution of the crisp subspace on α-level

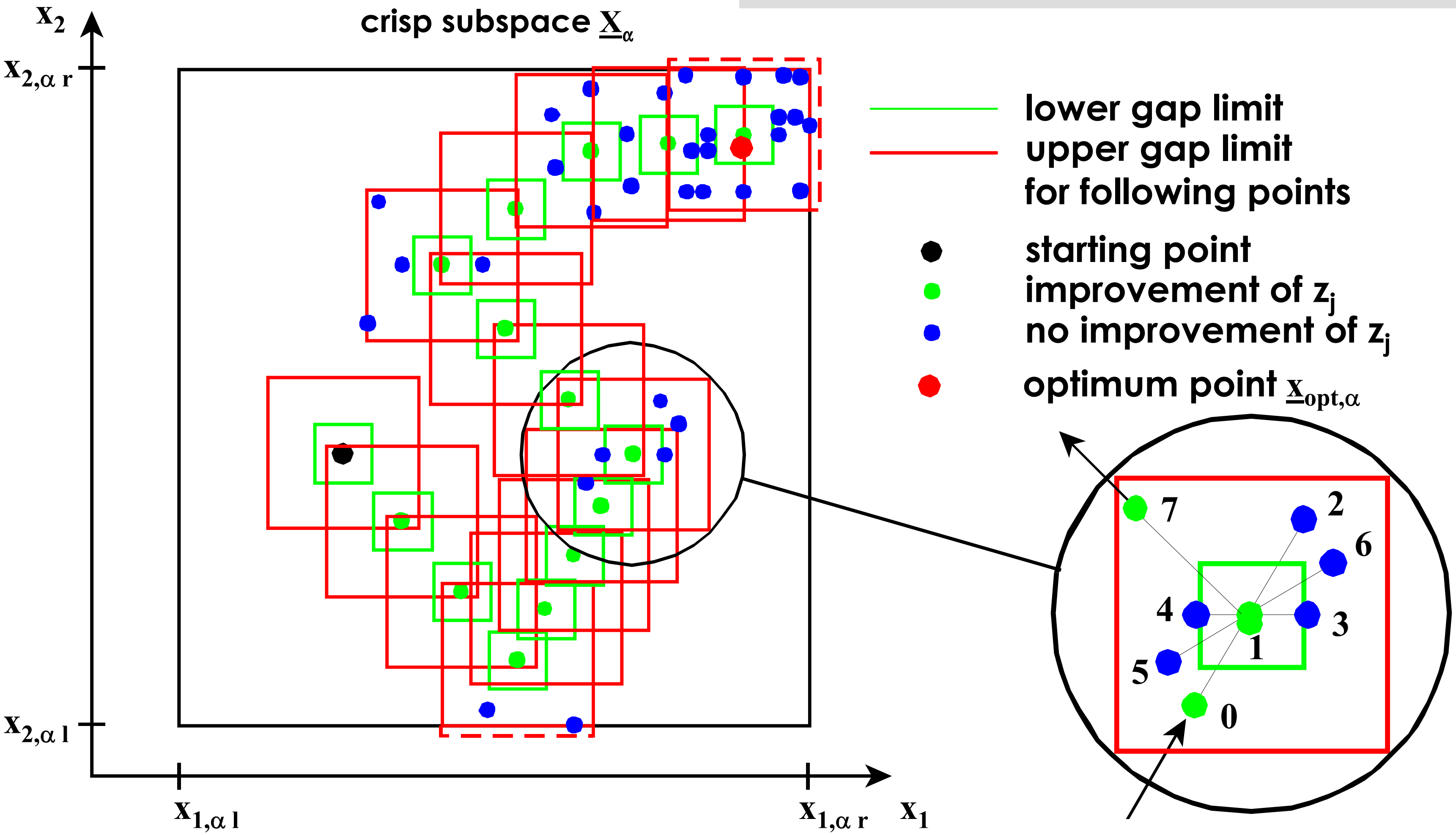
α



simultanious consideration of different input values



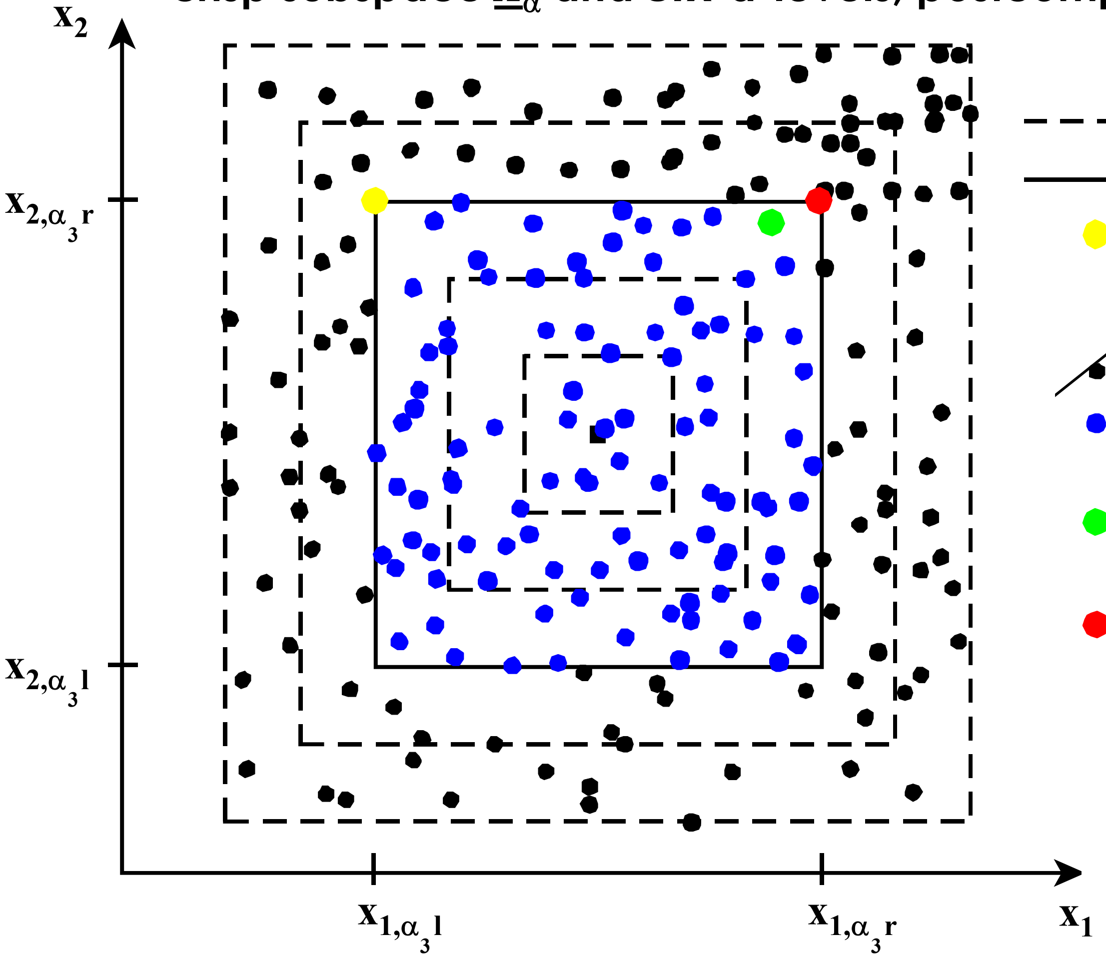
modified evolution strategy



Fuzzy analysis (7)

modified evolution strategy, post computing

crisp subspace $\underline{X}_\alpha$ and six α-levels, postcomputation for $\alpha = \alpha_3$



permissable domains for $\alpha \neq \alpha_3$

permissable domain for $\alpha = \alpha_3$

optimum point $\underline{x}_{opt,\alpha}$ für $\alpha = \alpha_3$ from α-level optimization

nonpermissable points for $\alpha = \alpha_3$

permissable points for $\alpha = \alpha_3$, point for rechecking

"best" point from post computation, new starting point

new optimum point for $\alpha = \alpha_3$

Properties of the mapping model

- Biuniqueness
- Uniquness
- Contiuity : *Uniqueness and contiuity lead to convex fuzzy results*

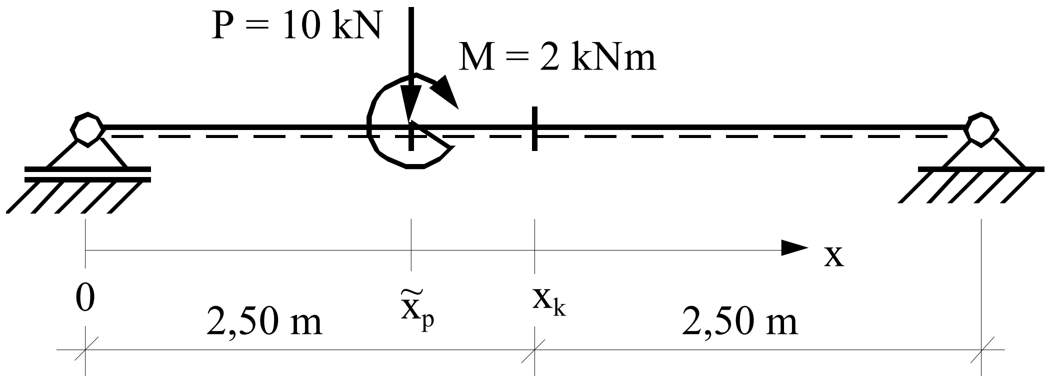
- Monotonicity: *The optimum point is located on the boundary of the input space*

- Dimensionality of the input and result space

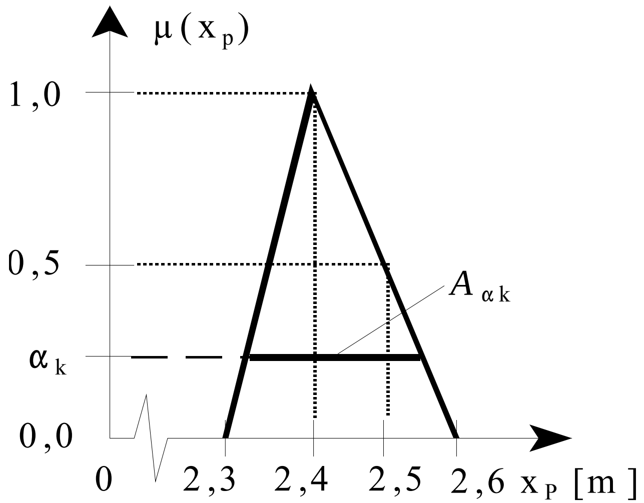
see file 1.2 for Example with non-monotonic mapping

Non-continuous mapping model

- **system**



- **Fuzzy input variable**



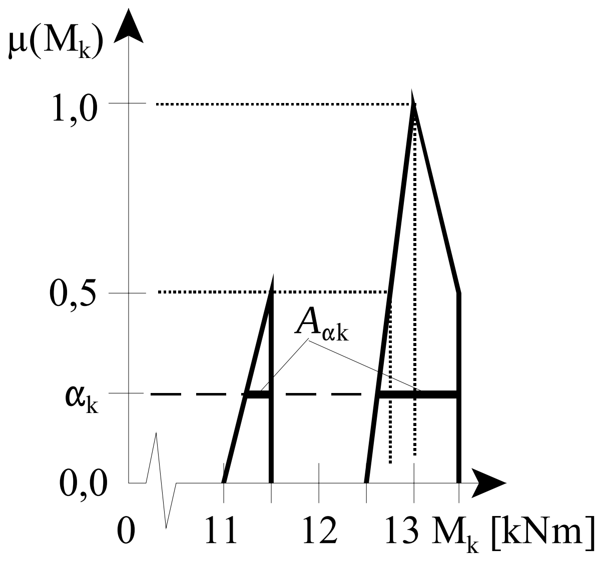
- **Mapping model**

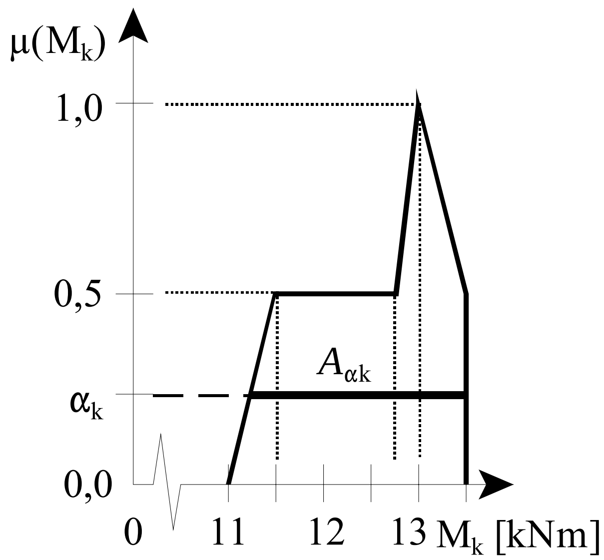
$$M_k(x_p) = \begin{cases} 1 + 5 \cdot x_p & [\text{kNm}] \mid x_p < x_k \\ 24 - 5 \cdot x_p & [\text{kNm}] \mid x_p > x_k \end{cases}$$

Noncontinuous mapping model

Fuzzy result variable $\tilde{M}_k$

Extension principle   α-level optimization

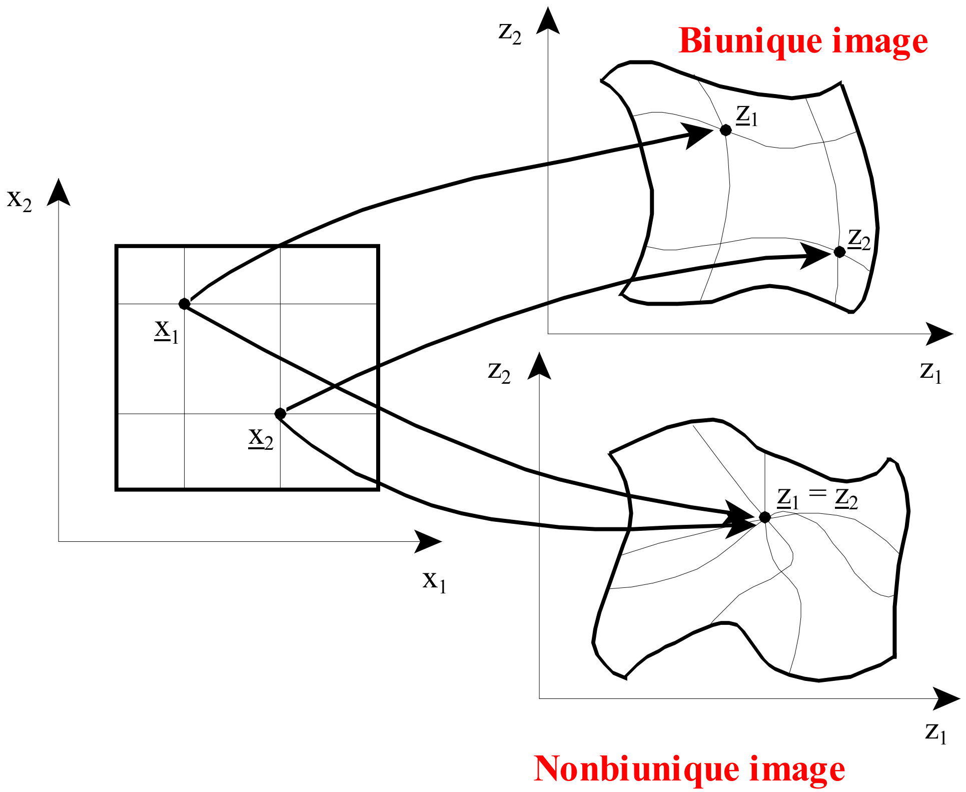




- a-level optimization produces convex membership function
- Extension principle produces disjointed fuzzy sets

Properties of the mapping model

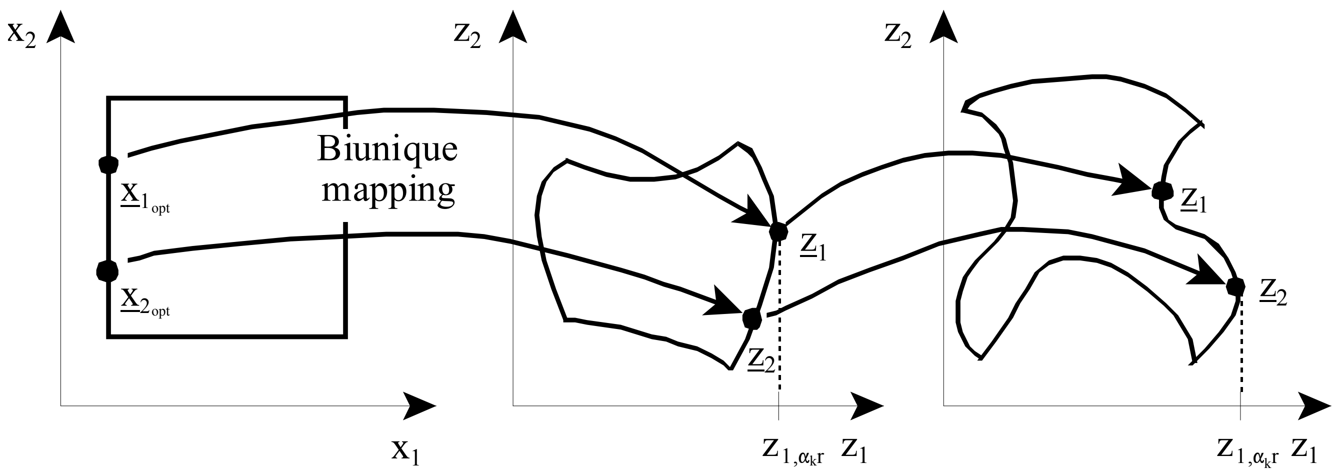
- **Biuniqueness**



Properties of the mapping model

Development of the z-space as a function of the crisp parameter t

$$z_j = f_j\left(x_1; \ldots; x_n; t_1; \ldots; t_p\right)$$

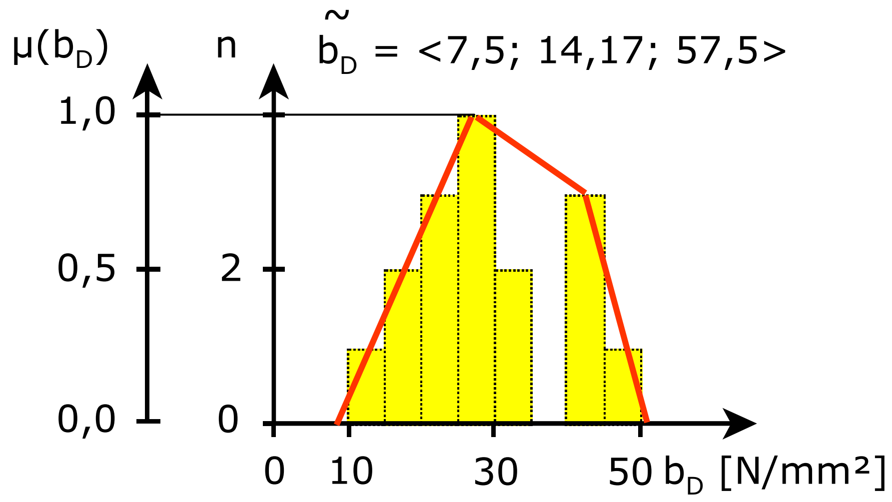


Changing of the crisp parameter t

Uncertainty quantification (1)

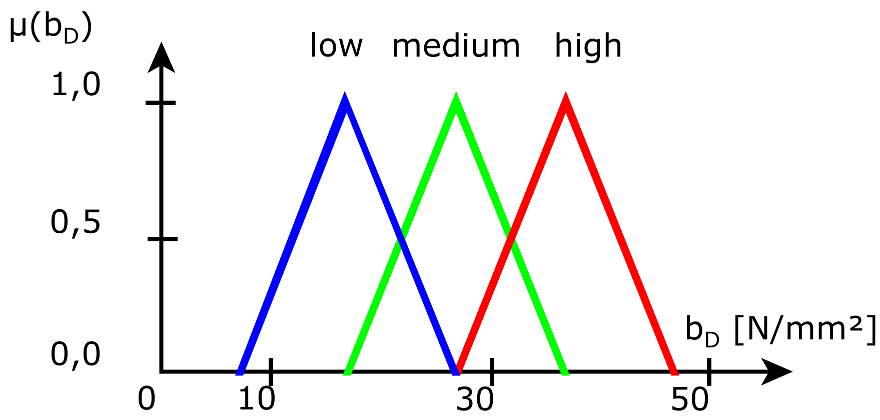
limited data basis

- sample with a slight number of sample elements



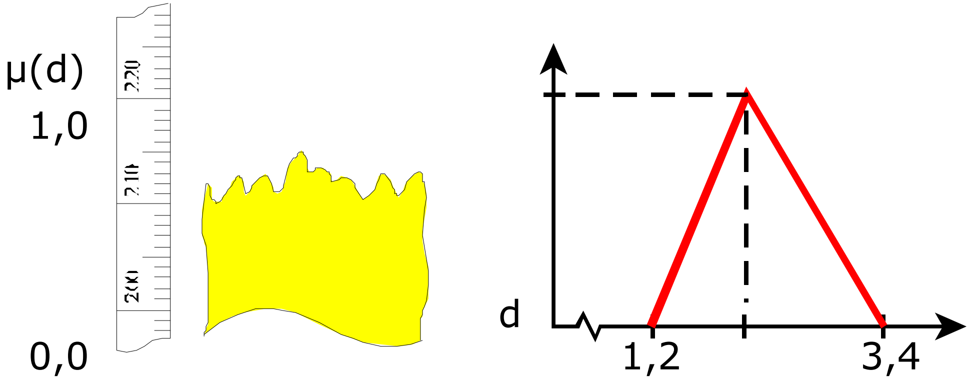
linguistic variable

- quality assessment

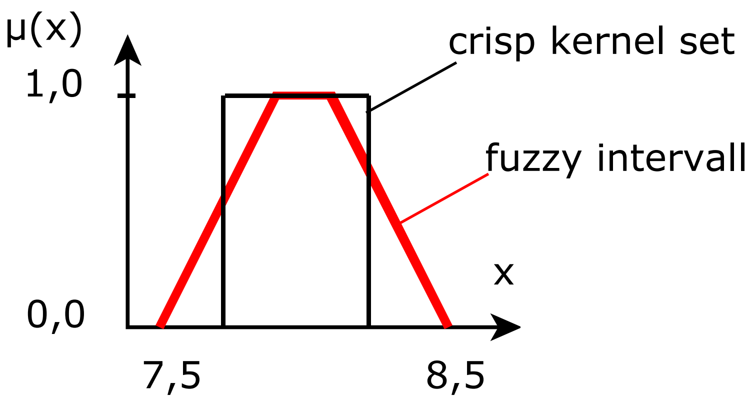


single uncertain measured value

- component thickness with rough surface

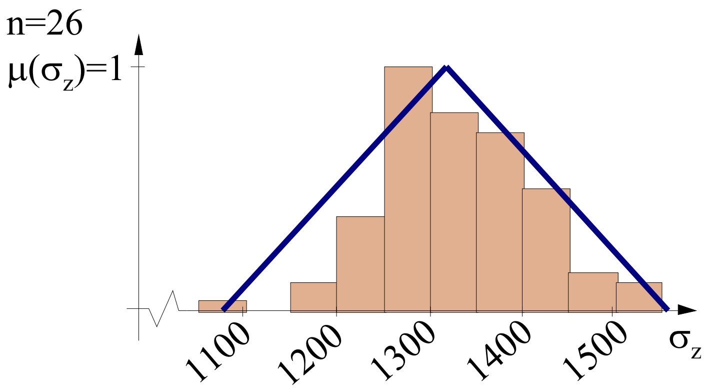


expert knowledge

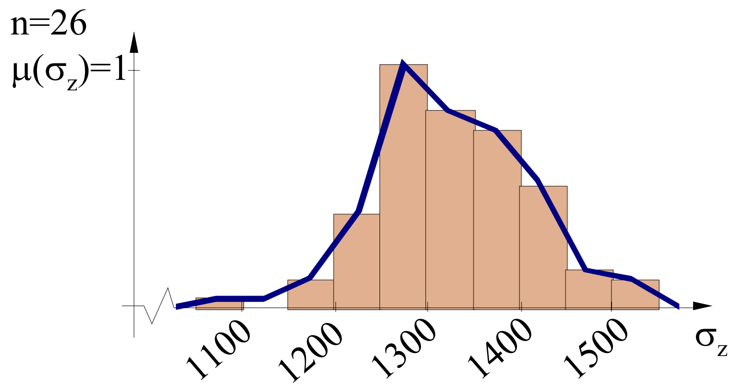


design of membership functions

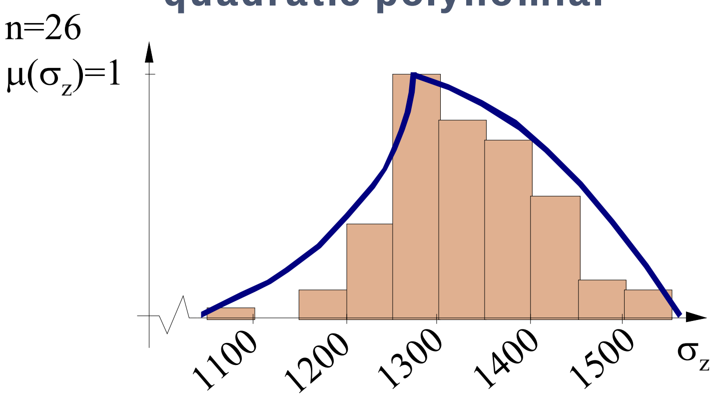
linear polynomial



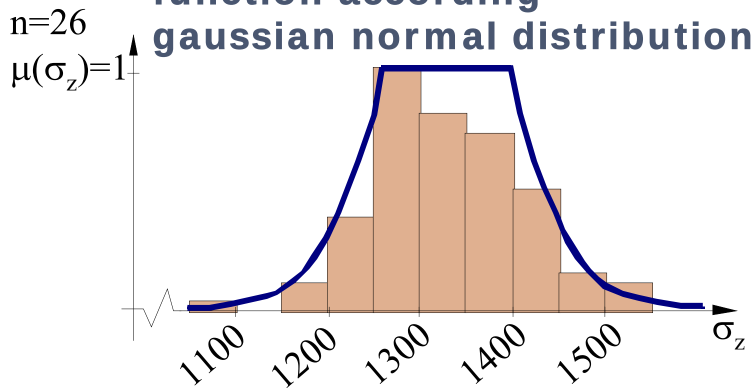
polynomial



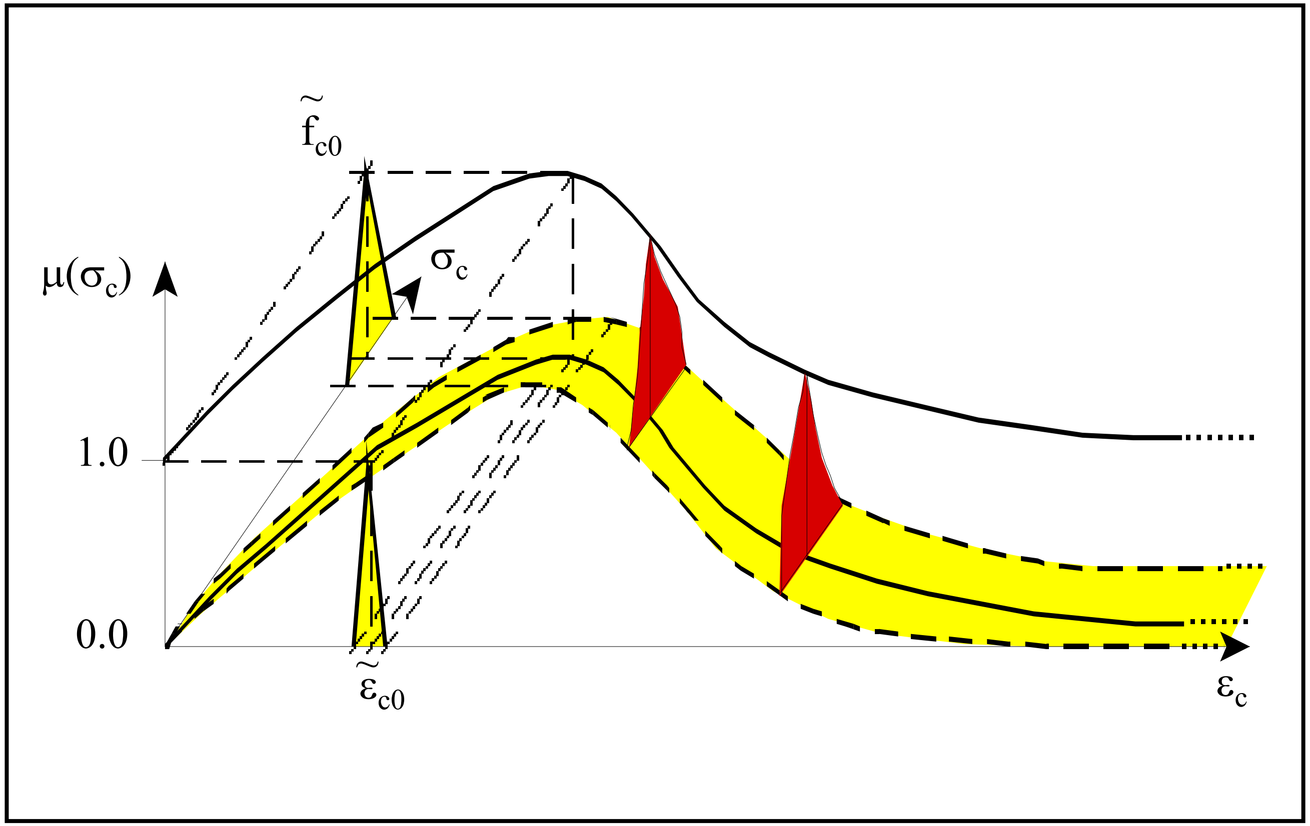
quadratic polynomial



function according gaussian normal distribution

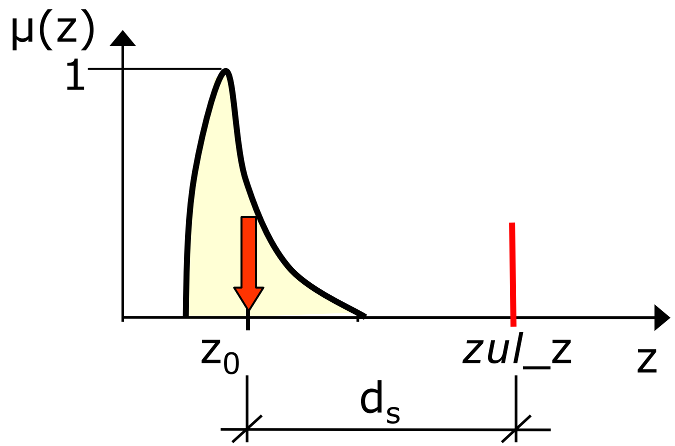


fuzzy function for the material behaviour of concrete



defuzzification of fuzzy values

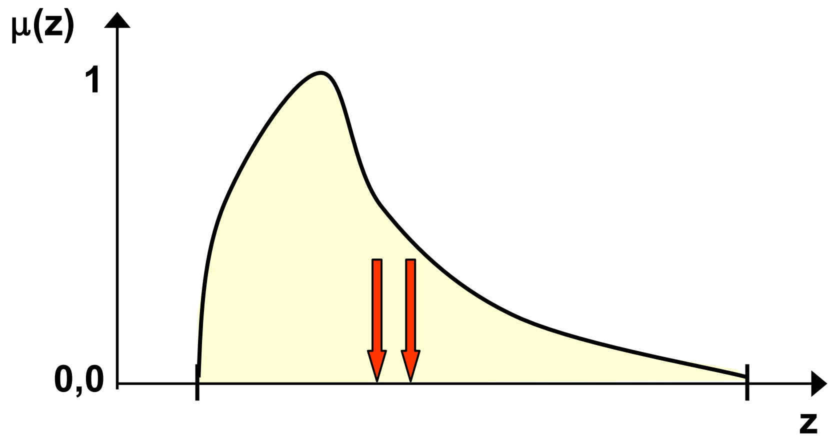
- conversion of a fuzzy value z into a crisp (deterministic) value



- disadvantage: loss of information

defuzzification methods

- Centroid method
- level rank method



Assessment of fuzzy results (3)

SHANNONs entropy

Comparison of the degree of uncertainty

$$H(\tilde{z}) = -k \cdot \int_{z \in \tilde{z}} \left[\mu(z) \cdot \ln(\mu(z)) + (1 - \mu(z)) \cdot \ln(1 - \mu(z)) \right] dz$$

- measure of the average information content
- evaluation of the fuzziness

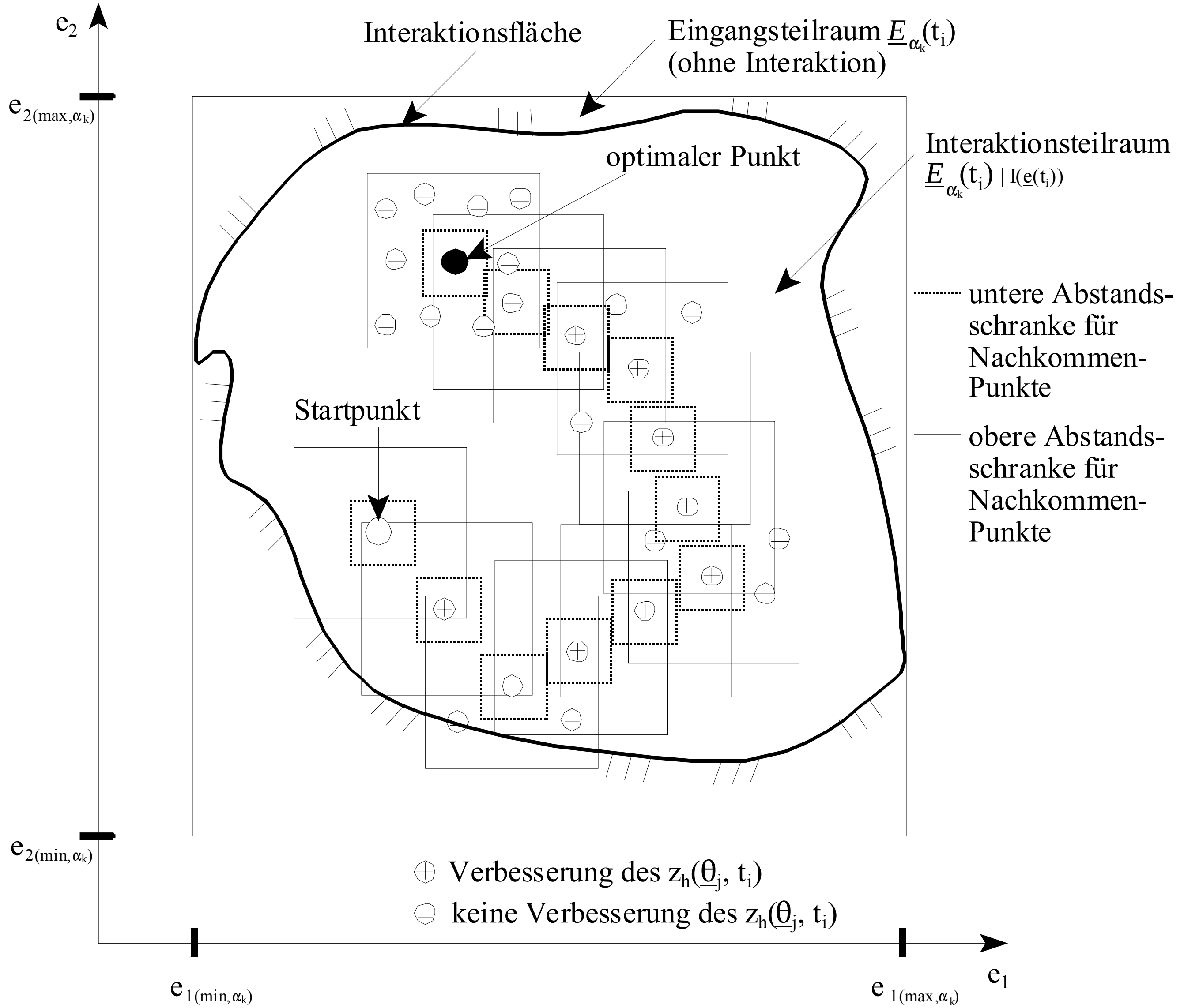
robustness measure

$$\text{robustness} = \frac{\text{uncertainty of input parameters}}{\text{uncertainty of the result parameters}}$$

$$\boxed{R(\tilde{z}_j) = \frac{1}{\sum_i \frac{H(\tilde{z}_j)}{H(\tilde{x}_i)}}}$$

see file 1.3 for
Comments to interaction between fuzzy variables

α-level optimization with interaction



Thank you !